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The intersection of essential approximate point spectra of operator matrices

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Abstract

When $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are given, we denote by M_C the operator acting on the infinite-dimensional separable Hilbert space $\mathcal{H} \oplus \mathcal{K}$ of the form $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. In this paper, it is shown that there exists some operator $C \in B(\mathcal{K}, \mathcal{H})$ such that M_C is upper semi-Fredholm and $\mathrm{ind}(M_C) \leqslant 0$ if and only if there exists some left invertible operator $C \in B(\mathcal{K}, \mathcal{H})$ such that M_C is upper semi-Fredholm and $\mathrm{ind}(M_C) \leqslant 0$. A necessary and sufficient condition for M_C to be upper semi-Fredholm and $\mathrm{ind}(M_C) \leqslant 0$ for some $C \in \mathrm{Inv}(\mathcal{K}, \mathcal{H})$ is given, where $\mathrm{Inv}(\mathcal{K}, \mathcal{H})$ denotes the set of all the invertible operators of $B(\mathcal{K}, \mathcal{H})$. In addition, we give a necessary and sufficient condition for M_C to be upper semi-Fredholm and $\mathrm{ind}(M_C) \leqslant 0$ for all $C \in \mathrm{Inv}(\mathcal{K}, \mathcal{H})$.

Keywords: Essential approximate point spectrum; 2×2 operator matrix; Perturbations of spectra

1. Introduction

The study of upper triangular operator matrices arises naturally from the following fact: if T is a Hilbert space operator and M is an invariant subspace for T, then T has the following 2×2 upper triangular operator matrix representation:

$$T = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} : \mathcal{M} \oplus \mathcal{M}^{\perp} \to \mathcal{M} \oplus \mathcal{M}^{\perp},$$

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and one way to study operator is to see them as entries of simpler operators. Recently, many authors have paid much attention to 2×2 upper triangular operator matrices (see [2–5,7,9,10]). For a given pair (A, B) of operators, Du and Pan (see [5]) give a necessary and sufficient condition for which M_C is invertible for some $C \in B(\mathcal{K}, \mathcal{H})$, Han et al. (see [9]) extended the result for operators A, B, C on Banach space. For the essential spectrum $\sigma_{\mathrm{e}}(T)$, the Weyl spectrum $\sigma_{\mathrm{w}}(T)$ and the Browder spectrum $\sigma_{\mathrm{b}}(T)$ of T, analogous results have been obtained in many literatures (see [2,3,5,7]).

Throughout this paper, let $\mathcal{H}$ and $\mathcal{K}$ be complex separable Hilbert spaces, let $B(\mathcal{H}, \mathcal{K})$, $B_{\mathrm{l}}(\mathcal{H}, \mathcal{K})$ and $\mathrm{Inv}(\mathcal{H}, \mathcal{K})$, respectively, denote the set of bounded linear operators, left invertible bounded linear operators and invertible bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, respectively, and abbreviate $B(\mathcal{H}, \mathcal{H})$ to $B(\mathcal{H})$. If $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ and $C \in B(\mathcal{K}, \mathcal{H})$, we define an operator M_C acting on $\mathcal{H} \oplus \mathcal{K}$ by the form

$$M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

For an operator T, we use $N(T)$ and $R(T)$ to denote the null space and the range of T, respectively. Let $n(T)$ be the nullity of T which is equal to $\dim N(T)$, and let $d(T)$ be the deficiency of T which is equal to $\dim N(T^*)$. An operator $T \in B(\mathcal{H}, \mathcal{K})$ (or $B(\mathcal{H})$) is said to be upper semi-Fredholm if $R(T)$ is closed and $N(T)$ has finite dimension and lower semi-Fredholm if $R(T)$ is closed and $N(T^*)$ has finite dimension. An operator T is called Fredholm if it is both upper semi-Fredholm and lower semi-Fredholm. Let $\Phi_+(H)$ ($\Phi_-(H)$) denotes the set of all upper (lower) semi-Fredholm operators. For an operator T, the left (right) essential spectrum $\sigma_{\mathrm{le}}(T)$ ($\sigma_{\mathrm{re}}(T)$) is defined by

$$\sigma_{\mathrm{le}}(T)\big(\sigma_{\mathrm{re}}(T)\big) = \big\{\lambda \in \mathbb{C}\colon\ T - \lambda \text{ is not upper (lower) semi-Fredholm}\big\}.$$

If T is a semi-Fredholm operator, we define the index of T by $\mathrm{ind}(T) = n(T) - d(T)$. An operator $T \in B(\mathcal{H}, \mathcal{K})$ is called Weyl if it is a Fredholm operator of index zero.

Let $\Phi_+^-(H)$ ($\Phi_+^-(H, K)$) (introduced in [11]) be the class of all $T \in \Phi_+(H)$ ($T \in \Phi_+(H, K)$) with $\mathrm{ind}(T) \leqslant 0$ for any $T \in B(\mathcal{H})$ ($T \in B(\mathcal{H}, \mathcal{K})$), let $\Phi_-^+(H)$ ($\Phi_-^+(H, K)$) be the class of all $T \in \Phi_-(H)$ ($T \in \Phi_-(H, K)$) with $\mathrm{ind}(T) \geqslant 0$ for any $T \in B(\mathcal{H})$ ($T \in B(\mathcal{H}, \mathcal{K})$), let

$$\begin{aligned} &\sigma_{\mathrm{ea}}(T) = \big\{\lambda \in \mathbb{C}\colon\ T - \lambda \text{ is not in } \Phi_+^-(H)\big\}, \\ &\sigma_{\mathrm{SF}_-^+}(T) = \big\{\lambda \in \mathbb{C}\colon\ T - \lambda \text{ is not in } \Phi_-^+(H)\big\}, \\ &\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{ea}}(T) \cup \sigma_{\mathrm{SF}_-^+}(T). \end{aligned}$$

Cao and Meng (in [2]) give a necessary and sufficient condition for which $M_C \in \Phi_+^-(H)$ for some $C \in B(\mathcal{K}, \mathcal{H})$ and characterize the set of $\bigcap_{C \in B(\mathcal{K}, \mathcal{H})} \sigma_{\mathrm{ea}} M_C$. In this paper, our main goal is to characterize the intersection of $\bigcap_{C \in B_{\mathrm{l}}(\mathcal{K}, \mathcal{H})} \sigma_{\mathrm{ea}}(M_C)$ and $\bigcap_{C \in \mathrm{Inv}(\mathcal{K}, \mathcal{H})} \sigma_{\mathrm{ea}}(M_C)$. This paper is organized as follows. In Section 2, we give a necessary and sufficient condition for which $M_C \in \Phi_+^-(H)$ for some $C \in B_{\mathrm{l}}(\mathcal{K}, \mathcal{H})$ and get

$$\bigcap_{C \in B_{\mathrm{l}}(\mathcal{K}, \mathcal{H})} \sigma_{\mathrm{ea}}(M_C) = \bigcap_{C \in B(\mathcal{K}, \mathcal{H})} \sigma_{\mathrm{ea}}(M_C).$$

In Section 3, we give a necessary and sufficient condition for which $M_C \in \Phi_+^-(H)$ for some $C \in \mathrm{Inv}(\mathcal{K}, \mathcal{H})$ and get

$$\bigcap_{C \in \mathrm{Inv}(\mathcal{K}, \mathcal{H})} \sigma_{\mathrm{ea}}(M_C) = \bigcap_{C \in B(\mathcal{K}, \mathcal{H})} \sigma_{\mathrm{ea}}(M_C) \cup \{\lambda \in \mathbb{C}\colon\ B - \lambda \text{ is compact}\}.$$

In Section 4, we give a necessary and sufficient condition for which $M_C \in \Phi_+^-(H)$ for all $C \in \mathrm{Inv}(\mathcal{K}, \mathcal{H})$. In addition, the idea in this paper is different from [2].

2. $\bigcap_{C \in B_l(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{ea}}(M_C)$

In order to prove our main results, we begin with some lemmas.

Lemma 2.1. *Let $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ and $C \in B(\mathcal{K}, \mathcal{H})$. If C as an operator from $N(B) \oplus N(B)^\perp$ into $R(A)^\perp \oplus \overline{R(A)}$ has the following operator matrix*:

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, \tag{1}$$

then

(a) *$M_C \in \Phi_+^-(H \oplus K)$ if and only if*
 (i) *$A \in \Phi_+(H)$;*
 (ii) *$M_1 \in \Phi_+^-(N(A) \oplus N(B) \oplus N(B)^\perp, R(A)^\perp \oplus R(B)^\perp \oplus \overline{R(B)})$ where*

$$M_1 := \begin{pmatrix} 0 & C_1 & C_2 \\ 0 & 0 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : N(A) \oplus N(B) \oplus N(B)^\perp \to R(A)^\perp \oplus R(B)^\perp \oplus \overline{R(B)}, \tag{2}$$

 where B as an operator from $N(B) \oplus N(B)^\perp$ into $R(B)^\perp \oplus \overline{R(B)}$ has the operator matrix $B = \left(\begin{smallmatrix} 0 & 0 \\ 0 & B_1 \end{smallmatrix}\right)$.

(b) *If $R(B)$ is closed, then $M_C \in \Phi_+^-(H \oplus K)$ if and only if*
 (i) *$A \in \Phi_+(H)$;*
 (ii) *C_1 is an operator with $R(C_1)$ is closed, $n(C_1) < \infty$, and $n(C_1) + n(A) \leqslant d(C_1) + d(B)$.*

Proof. (a) *Sufficiency*. Since $A \in \Phi_+(H)$, then $R(A)$ is closed. The space $\mathcal{H} \oplus \mathcal{K}$ can be decomposed as the following direct sums:

$$\mathcal{H} \oplus \mathcal{K} = N(A) \oplus N(A)^\perp \oplus N(B) \oplus N(B)^\perp = R(A)^\perp \oplus R(A) \oplus \overline{R(B)} \oplus R(B)^\perp.$$

Thus M_C as an operator from $N(A) \oplus N(A)^\perp \oplus N(B) \oplus N(B)^\perp$ into $R(A)^\perp \oplus R(A) \oplus \overline{R(B)} \oplus R(B)^\perp$ has the following operator matrix:

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} 0 & 0 & C_1 & C_2 \\ 0 & A_1 & C_3 & C_4 \\ 0 & 0 & 0 & B_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3}$$

where A_1 is an operator from $N(A)^\perp$ onto $R(A)$ and B_1 is an operator from $N(B)^\perp$ into $\overline{R(B)}$. By the assumption that $A \in \Phi_+(H)$, A_1 is an invertible operator. In this case, we have

$$\begin{pmatrix} 0 & 0 & C_1 & C_2 \\ 0 & A_1 & C_3 & C_4 \\ 0 & 0 & 0 & B_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & -A_1^{-1}C_3 & -A_1^{-1}C_4 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 & C_1 & C_2 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & 0 & B_1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & -A_1^{-1}C_3 & -A_1^{-1}C_4 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

is an invertible operator from $N(A) \oplus N(A)^{\perp} \oplus N(B) \oplus N(B)^{\perp}$ onto $N(A) \oplus N(A)^{\perp} \oplus N(B) \oplus N(B)^{\perp}$. Thus $M_C \in \Phi_+^-(H \oplus K)$ if and only if

$$\begin{pmatrix} 0 & 0 & C_1 & C_2 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 \end{pmatrix} \in \Phi_+^-\big(N(A) \oplus N(A)^{\perp} \oplus N(B) \oplus N(B)^{\perp}, R(A)^{\perp} \oplus R(A) \oplus R(B)^{\perp} \oplus \overline{R(B)}\big).$$

It follows that if $A \in \Phi_+(H)$ then $M_C \in \Phi_+^-(H \oplus K)$ if and only if

$$M_1 = \begin{pmatrix} 0 & C_1 & C_2 \\ 0 & 0 & 0 \\ 0 & 0 & B_1 \end{pmatrix} \in \Phi_+^-\big(N(A) \oplus N(B) \oplus N(B)^{\perp}, R(A)^{\perp} \oplus R(B)^{\perp} \oplus \overline{R(B)}\big).$$

Necessity. Clearly, $A \in \Phi_+(H)$. From the discussion above, it is not difficult to get (ii).

(b) If $R(B)$ is closed, then B_1 as an operator from $N(B)^{\perp}$ into $R(B)$ is invertible. Thus

$$\begin{pmatrix} I & 0 & -C_2B_1^{-1} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} 0 & C_1 & C_2 \\ 0 & 0 & 0 \\ 0 & 0 & B_1 \end{pmatrix} = \begin{pmatrix} 0 & C_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_1 \end{pmatrix}.$$

Since $n(A) < \infty$ and B_1 is invertible, we conclude that $M_1 \in \Phi_+^-(N(A) \oplus N(B) \oplus N(B)^{\perp}, R(A)^{\perp} \oplus R(B)^{\perp} \oplus \overline{R(B)})$ if and only if $R(C_1)$ is closed, $n(C_1) < \infty$ and $n(C_1) + n(A) \leqslant d(C_1) + d(B)$. $\Box$

Corollary 2.2. *Let (A, B) be a given pair of operators. If $A \in \Phi_+(H)$, $R(B)$ is closed and $d(A) + d(B) < n(A) + n(B)$, then for all $C \in B(\mathcal{K}, \mathcal{H})$, $M_C \notin \Phi_+^-(H \oplus K)$.*

Proof. Suppose that C has the operator matrix form (1) for all $C \in B(\mathcal{K}, \mathcal{H})$.

(i) $n(B) = \infty$. Since $d(A) < \infty$, then $n(C_1) = \infty$ for all C. By Lemma 2.1, $M_C \notin \Phi_+^-(H \oplus K)$.

(ii) $n(B) < \infty$, $n(A) < \infty$, $d(A) < \infty$ and $d(B) < \infty$. Since C_1 is an operator from $N(B)$ into $R(A)^{\perp}$, then

$$n(B) = n(C_1) + \dim N(C_1)^{\perp} \quad \text{and} \quad d(A) = d(C_1) + \dim R(C_1).$$

Thus $n(C_1) + n(A) > d(B) + d(C_1)$ since $n(B) + n(A) > d(A) + d(B)$ and $\dim N(C_1)^{\perp} = \dim R(C_1)$. From Lemma 2.1, $M_C \notin \Phi_+^-(H \oplus K)$ for all C. $\Box$

Corollary 2.3. *If $R(B)$ is closed, $A \in \Phi_+(H)$ and $n(B) + n(A) \leqslant d(B) + d(A)$, then $M_C \notin \Phi_+^-(H \oplus K)$ for any $C \in B_1(\mathcal{K}, \mathcal{H})$ if and only if $d(A) < \infty$ and $n(B) = d(B) = \infty$.*

Proof. Suppose that C has the operator matrix form (1).

Sufficiency is clear, since $n(B) = d(B) = \infty$ and $d(A) < \infty$, then $n(C_1) = \infty$. By Lemma 2.1, $M_C \notin \Phi_+^-(H \oplus K)$ for all C.

Necessity. Suppose that $d(A) < \infty$ and $n(B) = d(B) = \infty$ are not satisfied. There are four cases to consider.

Case 1. $n(B) = d(A) = \infty$.

Assume that $n(A) \leqslant d(B)$. Let S be an unitary operator from $N(B)$ onto $R(A)^\perp$. Since $A \in \Phi_+(H)$, $\dim R(A) = \infty$, let S_1 be a left invertible operator from $N(B)^\perp$ into $R(A)$. Define an operator C_0 by

$$C_0 = \begin{pmatrix} S & 0 \\ 0 & S_1 \end{pmatrix} : N(B) \oplus N(B)^\perp \to R(A)^\perp \oplus R(A)$$

then $M_{C_0} \in \Phi_+^-(H \oplus K)$ by Lemma 2.1.

If $n(A) > d(B)$ and $\{e_i\}_{i=1}^\infty$ and $\{f_i\}_{i=1}^\infty$ are orthogonal bases of $N(B)$ and $R(A)^\perp$, respectively, denote $n(A) - d(B) = m$, and define C_1 as an operator from $N(B)$ into $R(A)^\perp$ by

$$C_1(e_i) = f_{m+i}, \quad i = 1, 2, \ldots.$$

Clearly, $n(C_1) = 0$ and $n(C_1^*) = m$, then $n(C_1) + n(A) = d(C_1) + d(B)$. Define an operator C_0 by $C_0 = \left(\begin{smallmatrix} C_1 & 0 \\ 0 & S_1 \end{smallmatrix}\right)$. From Lemma 2.1, $M_{C_0} \in \Phi_+^-(H \oplus K)$.

Case 2. $n(B) < \infty$ and $d(A) = \infty$.

It is easy to show that $M_C \in \Phi_+^-(H \oplus K)$, for all $C \in B(\mathcal{K}, \mathcal{H})$.

Case 3. $n(B) < \infty$, $d(A) < \infty$ and $d(B) = \infty$.

It is clear that $M_C \in \Phi_+^-(H \oplus K)$, for all $C \in B(\mathcal{K}, \mathcal{H})$.

Case 4. $n(B) < \infty$, $d(A) < \infty$ and $d(B) < \infty$.

As the similar way with the proof of Corollary 2.2(ii), we can prove that $n(C_1) < \infty$ and $n(C_1) + n(A) \leqslant d(C_1) + d(B)$ for all C. $\square$

The following theorem is our main result in this section.

Theorem 2.4. *For a given pair (A, B) of operators, we have*

$$\bigcap_{C \in B_1(\mathcal{K}, \mathcal{H})} \sigma_{ea}(M_C) = \sigma_{le}(A) \cup \Phi_{lw}(A, B) \cup \Upsilon_{lw}(A, B) \cup \Psi_1(A, B),$$

where

$$\begin{aligned} \Psi_1(A, B) &= \{\lambda \in \mathbb{C}:\ R(B - \lambda) \text{ is not closed and } d(A - \lambda) < \infty\}, \\ \Phi_{lw}(A, B) &= \{\lambda \in \mathbb{C}:\ R(B - \lambda) \text{ is closed and} \\ &\qquad n(B - \lambda) + n(A - \lambda) > d(B - \lambda) + d(A - \lambda)\}, \\ \Upsilon_{lw}(A, B) &= \{\lambda \in \mathbb{C}:\ R(B - \lambda) \text{ is closed},\ n(B - \lambda) = d(B - \lambda) = \infty \\ &\qquad \text{and } d(A - \lambda) < \infty\}. \end{aligned}$$

Proof. For convenience, we divide the proof into two steps.

Step 1. If $\lambda \in \Psi_1(A,B) \setminus \sigma_{le}(A)$, then for all $C \in B(\mathcal{K},\mathcal{H})$, $M_C - \lambda \notin \Phi_+^-(H \oplus K)$.

Suppose that $M_C - \lambda$ has the operator matrix (3) and C has the operator matrix (1). By Lemma 2.1, for all $C \in B(\mathcal{K},\mathcal{H})$, $M_C - \lambda \notin \Phi_+^-(H \oplus K)$ if and only if

$$\begin{pmatrix} 0 & C_1 & C_2 \\ 0 & 0 & 0 \\ 0 & 0 & B_1-\lambda \end{pmatrix} \notin \Phi_+^-\big(N(A-\lambda) \oplus N(B-\lambda) \oplus N(B-\lambda)^\perp, R(A-\lambda)^\perp \oplus R(B-\lambda)^\perp \oplus \overline{R(B-\lambda)}\big),$$

for all $C_1 \in B(N(B-\lambda), R(A-\lambda)^\perp)$, $C_2 \in B(N(B-\lambda)^\perp, R(A-\lambda)^\perp)$.

Conversely, assume that there exist $C_1^0 \in B(N(B-\lambda), R(A-\lambda)^\perp)$ and $C_2^0 \in B(N(B-\lambda)^\perp, R(A-\lambda)^\perp)$ such that

$$\begin{pmatrix} 0 & C_1 & C_2 \\ 0 & 0 & 0 \\ 0 & 0 & B_1-\lambda \end{pmatrix} \in \Phi_+^-\big(N(A) \oplus N(B) \oplus N(B)^\perp, R(A)^\perp \oplus R(B)^\perp \oplus \overline{R(B)}\big).$$

Then it is upper semi-Fredholm. By the assumption that $\lambda \in \Psi_1(A,B) \setminus \sigma_{le}(A)$, we have $d(A-\lambda) < \infty$. It follows that C_1^0 and C_2^0 are compact operators. Using 3.11 in Chapter XI of [1], we conclude that

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_1-\lambda \end{pmatrix}$$

is upper semi-Fredholm. Thus $R(B_1-\lambda)$ is closed. But $\lambda \in \Psi_1(A,B) \setminus \sigma_{le}(A)$ implies that $R(B-\lambda)$ is not closed. This is a contradiction.

Step 2. If $\lambda \in \{\lambda \in \mathbb{C}\colon\ R(B-\lambda) \text{ is not closed},\ d(A-\lambda)=\infty\} \setminus \sigma_{le}(A)$, then there exists $C_0 \in B_l(\mathcal{K},\mathcal{H})$, such that $M_{C_0} - \lambda \in \Phi_+^-(H \oplus K)$.

Let H_1 be a closed subspace of $R(A-\lambda)^\perp$ with $\dim H_1 = n(B-\lambda)$ and $\dim(R(A-\lambda)^\perp \ominus H_1) = \dim N(B-\lambda)^\perp$. Let C_1 and C_2 be unitary operators from $N(B-\lambda)$ onto H_1 and from $N(B-\lambda)^\perp$ onto $R(A-\lambda)^\perp \ominus H_1$, respectively. Define

$$C_0 = \begin{pmatrix} C_1 & C_2 \\ 0 & 0 \end{pmatrix} : N(B-\lambda) \oplus N(B-\lambda)^\perp \to R(A-\lambda)^\perp \oplus R(A-\lambda).$$

Clearly,

$$\begin{pmatrix} C_1^* & 0 \\ C_2^* & 0 \end{pmatrix}\begin{pmatrix} C_1 & C_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

where

$$\begin{pmatrix} C_1^* & 0 \\ C_2^* & 0 \end{pmatrix} : R(A-\lambda)^\perp \oplus R(A-\lambda) \to N(B-\lambda) \oplus N(B-\lambda)^\perp.$$

Thus C_0 is left invertible. Since

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -(B_1-\lambda)C_2^* & 0 & I \end{pmatrix}\begin{pmatrix} 0 & C_1 & C_2 \\ 0 & 0 & 0 \\ 0 & 0 & B_1-\lambda \end{pmatrix} = \begin{pmatrix} 0 & C_1 & C_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & C_1 & C_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\in \Phi_+^-\bigl(N(A-\lambda) \oplus N(B-\lambda) \oplus N(B-\lambda)^\perp, R(A-\lambda)^\perp \oplus R(B-\lambda)^\perp \oplus \overline{R(B-\lambda)}\bigr),$$

$M_{C_0} - \lambda \in \Phi_+^-(H \oplus K)$, by Lemma 2.1(a).

Finally, by Step 1, we can conclude that

$$\bigcap_{C\in B_l(\mathcal{K},\mathcal{H})} \sigma_{ea}(M_C) \supseteq \Psi_l(A,B) \setminus \sigma_{le}(A).$$

By Corollaries 2.2 and 2.3, it is easy to see that

$$\begin{aligned}\bigcap_{C\in B_l(\mathcal{K},\mathcal{H})} \sigma_{ea}(M_C) &\supseteq \bigl(\Psi_l(A,B) \setminus \sigma_{le}(A)\bigr) \cup \sigma_{le}(A) \cup \Phi_{lw}(A,B) \cup \Upsilon_{lw}(A,B) \\ &= \sigma_{le}(A) \cup \Phi_{lw}(A,B) \cup \Upsilon_{lw}(A,B) \cup \Psi_l(A,B).\end{aligned}$$

By Corollary 2.3 and Step 2, we get that

$$\bigcap_{C\in B_l(\mathcal{K},\mathcal{H})} \sigma_{ea}(M_C) \subseteq \sigma_{le}(A) \cup \Phi_{lw}(A,B) \cup \Upsilon_{lw}(A,B) \cup \Psi_l(A,B).$$

Combining the two inclusions above, we obtain

$$\bigcap_{C\in B_l(\mathcal{K},\mathcal{H})} \sigma_{ea}(M_C) = \sigma_{le}(A) \cup \Phi_{lw}(A,B) \cup \Upsilon_{lw}(A,B) \cup \Psi_l(A,B). \qquad \square$$

The following corollaries are immediate from Theorem 2.4.

Corollary 2.5. (See [2].) *For given $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$, we have*

$$\bigcap_{C\in B(\mathcal{K},\mathcal{H})} \sigma_{ea}(M_C) = \sigma_{le}(A) \cup \Phi_{lw}(A,B) \cup \Upsilon_{lw}(A,B) \cup \Psi_l(A,B).$$

Corollary 2.6. (See [2].) *For a given pair (A,B) of operators, we have*

$$\begin{aligned}\bigcap_{C\in B_r(\mathcal{K},\mathcal{H})} \sigma_{SF_-^+}(M_C) &= \bigcap_{C\in B(\mathcal{K},\mathcal{H})} \sigma_{SF_-^+}(M_C) \\ &= \sigma_{re}(B) \cup \Phi_{rw}(A,B) \cup \Upsilon_{rw}(A,B) \cup \Psi_r(A,B),\end{aligned}$$

where

$$\begin{aligned}\Phi_{rw}(A,B) &= \bigl\{\lambda \in \mathbb{C}\colon\ R(A-\lambda) \text{ is closed}, \\ &\qquad n(B-\lambda) + n(A-\lambda) < d(B-\lambda) + d(A-\lambda)\bigr\}, \\ \Upsilon_{rw}(A,B) &= \bigl\{\lambda \in \mathbb{C}\colon\ R(A-\lambda) \text{ is closed},\ n(A-\lambda) = d(A-\lambda) = \infty, n(B-\lambda) < \infty\bigr\}, \\ \Psi_r(A,B) &= \bigl\{\lambda \in \mathbb{C}\colon\ R(A-\lambda) \text{ is closed},\ n(B-\lambda) < \infty\bigr\}.\end{aligned}$$

It is a natural question that whether the equation

$$\bigcap_{C\in \mathrm{Inv}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{ea}}(M_C) = \bigcap_{C\in B(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{ea}}(M_C)$$

holds?

3. $\bigcap_{C\in \mathrm{Inv}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{ea}}(M_C)$

In this section, our main result is:

Theorem 3.1. *For given pair of operators* (A, B), *we have*

$$\bigcap_{C\in \mathrm{Inv}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{ea}}(M_C) = \bigcap_{C\in B(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{ea}}(M_C) \cup \{\lambda \in \mathbb{C}\colon\ B-\lambda \text{ is compact}\}.$$

We need the following lemmas.

Lemma 3.2. *Let* $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ *and* $C \in B(\mathcal{K},\mathcal{H})$. *If* C *has the operator matrix* (1), *then* M_C *is invertible if and only if* A *is left invertible,* B *is right invertible and* C_1 *is invertible.*

Proof. *Sufficiency*. Since A is left invertible, A_1 is invertible. Then

$$\begin{pmatrix} 0 & C_1 & C_2\\ A_1 & C_3 & C_4\\ 0 & 0 & B_1\end{pmatrix}\begin{pmatrix} I & -A_1^{-1}C_3 & -A_1^{-1}C_4\\ 0 & I & 0\\ 0 & 0 & I\end{pmatrix} = \begin{pmatrix} 0 & C_1 & C_2\\ A_1 & 0 & 0\\ 0 & 0 & B_1\end{pmatrix}.$$

Since B is right invertible, B_1 is invertible. Then

$$\begin{pmatrix} I & 0 & -C_2B_1^{-1}\\ 0 & I & 0\\ 0 & 0 & I\end{pmatrix}\begin{pmatrix} 0 & C_1 & C_2\\ A_1 & 0 & 0\\ 0 & 0 & B_1\end{pmatrix} = \begin{pmatrix} 0 & C_1 & 0\\ A_1 & 0 & 0\\ 0 & 0 & B_1\end{pmatrix}.$$

Therefore, if C_1 is invertible, then M_C is invertible.

Necessity. If M_C is invertible, then A is left invertible, B is right invertible. By the proof of sufficiency, we have that C_1 is invertible. □

Lemma 3.3. *If* $A \in \Phi_+(H)$, $n(B)+n(A) \leqslant d(A)+d(B)$ *and* $R(B)$ *is closed, then* $M_C \notin \Phi_+^-(H)$ *for any* $C \in \mathrm{Inv}(\mathcal{K},\mathcal{H})$ *if and only if one of the following conditions holds*:

(i) $\dim N(B)^\perp < \infty$,
(ii) $n(B) = d(B) = \infty$ *and* $d(A) < \infty$.

Proof. *Necessity*. Assume that (ii) is not satisfied. To show that (i) holds, we will prove that $\dim N(B)^\perp = \infty$ then there exist some $C \in \mathrm{Inv}(\mathcal{K},\mathcal{H})$ such that $M_C \in \Phi_+^-(H \oplus K)$. By Corollary 2.3, we only need to show that if $n(B) = d(A) = \infty$ and $\dim N(B)^\perp = \infty$ then there exist some $C \in \mathrm{Inv}(\mathcal{K},\mathcal{H})$ such that $M_C \in \Phi_+^-(H \oplus K)$.

Case 1. $n(A) \leqslant d(B)$. Let S be an unitary operator from $N(B)$ onto $R(A)^\perp$ and S_1 an invertible operator from $N(B)^\perp$ onto $R(A)$, since $\dim R(A) = \infty$. Set

$$C_0 = \begin{pmatrix} S & 0\\ 0 & S_1\end{pmatrix} : N(B) \oplus N(B)^\perp \to R(A)^\perp \oplus R(A)$$

then $M_{C_0} \in \Phi_+^-(H \oplus K)$ by Lemma 2.1.

Case 2. $n(A) > d(B)$. Denote $n(A) - d(B) = n$.

Let w_1 be a left invertible operator from $N(B)$ into $R(A)^\perp$ with $n(w_1^*) = 2n$, w_3 be a right invertible operator from $N(B)^\perp$ into $R(A)$ with $n(w_3) = 2n$, and w_2 be an invertible operator from $N(B)^\perp$ into $R(A)^\perp$ such that $P_{N(w_1^*)} w_2|_{N(w_3)}$ is an invertible operator from $N(w_3)$ onto $N(w_1^*)$, respectively, where $P_{N(w_1^*)}$ is the orthogonal projection onto $N(w_1^*)$. Clearly, $n(w_1) + n(A) \leqslant d(w_1) + d(B)$. Set

$$C_0 = \begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix} : N(B) \oplus N(B)^\perp \to R(A)^\perp \oplus R(A).$$

It is easy to see that C_0 is invertible, by Lemma 3.2. From Lemma 2.1, $M_{C_0} \in \Phi_+^-(H \oplus K)$.

Sufficiency. By Corollary 2.3, we only need to show that if $\dim N(B)^\perp < \infty$, then $M_C \notin \Phi_+^-(H)$ for all $C \in \mathrm{Inv}(\mathcal{K}, \mathcal{H})$. Since $A \in \Phi_+(H)$, $\dim R(A) = \infty$. By the contrary, assume that $M_C \in \Phi_+^-(H \oplus K)$, where $C \in \mathrm{Inv}(\mathcal{K}, \mathcal{H})$. We have that C_1 is an operator with $R(C_1)$ is closed, $n(C_1) < \infty$, by Lemma 2.1. Suppose that C_1^+ is an operator from $R(A)^\perp$ into $N(B)$ such that $C_1^+ C_1 = I_{N(B)} + K_0$ (see [8], Atkinson's theorem), where K_0 is a compact operator from $N(B)$ into $N(B)$. Thus

$$\begin{pmatrix} I & 0 \\ -C_3 C_1^+ & I \end{pmatrix} \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \\ -C_3 K_0 & C_4 - C_3 C_1^+ C_2 \end{pmatrix}$$

is invertible. Using 3.11 in Chapter XI of [1], we get that

$$\begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} : N(B) \oplus N(B)^\perp \to R(A)^\perp \oplus R(A)$$

is Fredholm. But this is a contradiction with the fact that $\dim R(A) = \infty$. □

Lemma 3.4. *If $A \in \Phi_+(H)$ and B is compact, then for all $C \in \mathrm{Inv}(\mathcal{K}, \mathcal{H})$, $M_C \notin \Phi_+^-(H \oplus K)$.*

Proof. Suppose, contrary to the assertion, that $M_{C_0} \in \Phi_+^-(H \oplus K)$, for some $C_0 \in \mathrm{Inv}(\mathcal{K}, \mathcal{H})$.

$$\begin{pmatrix} I & 0 \\ -BC_0^{-1} & I \end{pmatrix} \begin{pmatrix} A & C_0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & 0 \\ -C_0^{-1}A & I \end{pmatrix} = \begin{pmatrix} 0 & C_0 \\ -BC_0^{-1}A & 0 \end{pmatrix},$$

then $-BC_0^{-1}A \in \Phi_+(H, K)$. This is a contradiction with compactness of $-BC_0^{-1}A$. Hence, for all $C \in \mathrm{Inv}(\mathcal{K}, \mathcal{H})$, $M_C \notin \Phi_+^-(H \oplus K)$. □

Lemma 3.5. [6] *Let V be a linear subspace of $\mathcal{H}$. These are equivalent*:

(1) *Any bounded operator A on $\mathcal{H}$ with $R(A) \subseteq V$ is compact*;
(2) *V contains no closed infinite-dimensional subspace.*

Lemma 3.6. *If $A \in \Phi_+(H)$, $R(B)$ is not closed and $d(A) = \infty$, then B is not compact if and only if there exists $C \in \mathrm{Inv}(\mathcal{K}, \mathcal{H})$ such that $M_C \in \Phi_+^-(H \oplus K)$.*

Proof. *Sufficiency*. If B is compact, by Lemma 3.4, $M_C \notin \Phi_+^-(H \oplus K)$, for any $C \in \mathrm{Inv}(\mathcal{K}, \mathcal{H})$.

Necessity. If B is not compact, by Lemma 3.5, $R(B)$ contains a closed infinite-dimensional subspace. No loss of generality, suppose that K_1 is closed subspace of $R(B)$ with $\dim K_1 = \infty$ and $\dim K_1^\perp = \infty$. Let $H_1 = \{x \in N(B)^\perp\colon\ Bx \in K_1\}$. Thus H_1 is a closed subspace of

$N(B)^\perp$ and $\dim H_1 = \infty$. Denote $H_1^\perp = N(B)^\perp \ominus H_1$. No loss of generality, we may assume that $\dim H_1^\perp = \infty$. (Otherwise, suppose that $\{e_n\}_{n=1}^\infty$ is an orthonormal basis of H_1. Denote $H_0 = \operatorname{span}\{e_n\colon n = 2, 4, 6, \ldots\}$ and $K_0 = \{Bx\colon x \in H_0\}$, then H_1 and K_1 can be instead by H_0 and K_0, respectively.) Since $d(A) = \infty$, let $R(A)^\perp = H_2 \oplus H_2^\perp$ with $\dim H_2 = \dim N(B)$ and $\dim H_2^\perp = \infty$. Define an operator $C : \mathcal{K} \to \mathcal{H}$ by

$$C = \begin{pmatrix} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & V_3 \end{pmatrix} : N(B) \oplus H_1^\perp \oplus H_1 \to H_2 \oplus H_2^\perp \oplus R(A),$$

where V_1, V_2 and V_3 are unitary operators. Obviously, C is invertible. Suppose that $B_1 = B|_{N(B)^\perp}$, then

$$B_1 = \begin{pmatrix} B_{11} & B_{12} \\ B_{13} & 0 \end{pmatrix} : H_1^\perp \oplus H_1 \to K_1 \oplus K_1^\perp,$$

where B_{12} is an invertible operator from H_1 onto K_1. Hence M_1 (as Lemma 2.1) has the following operator matrix form:

$$M_1 = \begin{pmatrix} 0 & V_1 & 0 & 0 \\ 0 & 0 & V_2 & 0 \\ 0 & 0 & B_{11} & B_{12} \\ 0 & 0 & B_{13} & 0 \end{pmatrix} : N(A) \oplus N(B) \oplus H_1^\perp \oplus H_1 \to H_2 \oplus H_2^\perp \oplus K_1 \oplus K_1^\perp.$$

Let

$$W = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & -B_{11}V_2^* & I & 0 \\ 0 & -B_{13}V_2^* & 0 & I \end{pmatrix} : H_2 \oplus H_2^\perp \oplus K_1 \oplus K_1^\perp \to H_2 \oplus H_2^\perp \oplus K_1 \oplus K_1^\perp.$$

Then

$$WM_1 = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & -B_{11}V_2^* & I & 0 \\ 0 & -B_{13}V_2^* & 0 & I \end{pmatrix} \begin{pmatrix} 0 & V_1 & 0 & 0 \\ 0 & 0 & V_2 & 0 \\ 0 & 0 & B_{11} & B_{12} \\ 0 & 0 & B_{13} & 0 \end{pmatrix} = \begin{pmatrix} 0 & V_1 & 0 & 0 \\ 0 & 0 & V_2 & 0 \\ 0 & 0 & 0 & B_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to show that

$$\begin{pmatrix} 0 & V_1 & 0 & 0 \\ 0 & 0 & V_2 & 0 \\ 0 & 0 & 0 & B_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \Phi_+^-\big(N(A) \oplus N(B) \oplus H_1^\perp \oplus H_1, H_2 \oplus H_2^\perp \oplus K_1 \oplus K_1^\perp\big).$$

Therefore, $M_1 \in \Phi_+^-(N(A) \oplus N(B) \oplus H_1^\perp \oplus H_1, H_2 \oplus H_2^\perp \oplus K_1 \oplus K_1^\perp)$. By Lemma 2.1, $M_C \in \Phi_+^-(H \oplus K)$. $\square$

Proof of Theorem 3.1. By Lemma 3.4, it is clear that

$$\bigcap_{C \in \operatorname{Inv}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{ea}}(M_C) \supseteq \bigcap_{C \in B(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{ea}}(M_C) \cup \{\lambda \in \mathbb{C}\colon B - \lambda \text{ is compact}\}.$$

For the converse, let $\lambda \notin (\bigcap_{C \in B(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{ea}}(M_C) \cup \{\lambda \in \mathbb{C}\colon B - \lambda \text{ is compact}\})$.

Case 1. $R(B - \lambda)$ is not closed. Then $d(A - \lambda) = \infty$, $A - \lambda \in \Phi_+(H)$ and $B - \lambda$ is not compact. By Lemma 3.6, there exists $C \in \operatorname{Inv}(\mathcal{K}, \mathcal{H})$ such that $M_C - \lambda \in \Phi_+^-(H \oplus K)$.

Case 2. $R(B-\lambda)$ is closed. Since $B-\lambda$ is not compact, then $\dim N(B-\lambda)^{\perp}=\infty$. By Lemma 3.3, there exists $C\in \mathrm{Inv}(\mathcal{K},\mathcal{H})$ such that $M_C-\lambda\in \Phi_+^-(H\oplus K)$, since $A-\lambda\in\Phi_+(H)$ and $d(A-\lambda)+d(B-\lambda)\geqslant n(A-\lambda)+n(B-\lambda)$. $\square$

In the similar way, we have

Corollary 3.7. *For a given pair of operators* (A,B), *we have*

$$\bigcap_{C\in \mathrm{Inv}(\mathcal{K},\mathcal{H})}\sigma_{\mathrm{SF}_-^+}(M_C)=\bigcap_{C\in B(\mathcal{K},\mathcal{H})}\sigma_{\mathrm{SF}_-^+}(M_C)\cup\{\lambda\in\mathbb{C}\colon\ A-\lambda \text{ is compact}\}.$$

Theorem 3.8. *For a given pair of operators* (A,B), *we have*

$$\bigcap_{C\in \mathrm{Inv}(\mathcal{K},\mathcal{H})}\sigma_{\mathrm{w}}(M_C)=\bigcap_{C\in B(\mathcal{K},\mathcal{H})}\sigma_{\mathrm{w}}(M_C)\cup\{\lambda\in\mathbb{C}\colon\ A-\lambda \text{ or } B-\lambda \text{ is compact}\}.$$

4. $\bigcup_{C\in \mathrm{Inv}(\mathcal{K},\mathcal{H})}\sigma_{\mathrm{ea}}(M_C)$

Theorem 4.1. *For a given pair of operators* (A,B), $M_C\in\Phi_+^-(H\oplus K)$ *for all* $C\in\mathrm{Inv}(\mathcal{K},\mathcal{H})$ *if and only if the following conditions hold*:

(i) $A\in\Phi_+(H)$;
(ii) $B\in\Phi_+(H)$;
(iii) $\mathrm{ind}(A)+\mathrm{ind}(B)\leqslant 0$.

Proof. *Sufficiency* is clear, since $M_C=\begin{pmatrix} I & 0\\ 0 & B\end{pmatrix}\begin{pmatrix} I & C\\ 0 & I\end{pmatrix}\begin{pmatrix} A & 0\\ 0 & I\end{pmatrix}$.

Necessity. It is clear that $A\in\Phi_+(H)$ and $n(B)<\infty$. We firstly show that $R(B)$ is closed. Assume to the contrary that $R(B)$ is not closed. By Theorem 2.4 and Lemma 3.4, $d(A)=\infty$ and B is not compact. Thus $R(B)$ contains a closed infinite-dimensional subspace K_1 with $\dim K_1^{\perp}=\infty$. Let $H_1=\{x\in N(B)^{\perp}\colon\ Bx\in K_1\}$. Using the same technique as Lemma 3.6, we may assume that H_1 is a closed subspace of $N(B)^{\perp}$, $\dim H_1=\infty$ and $\dim H_1^{\perp}=\infty$. Since $d(A)=\infty$, set $R(A)^{\perp}=H_2\oplus H_2^{\perp}$ with $\dim H_2=\dim N(B)$ and $\dim H_2^{\perp}=\infty$. Thus

$$M_C=\begin{pmatrix} 0 & 0 & C_{11} & C_{12}\\ 0 & A_1 & C_{21} & C_{22}\\ 0 & 0 & B_1 & B_2\\ 0 & 0 & 0 & B_4\end{pmatrix}: N(A)\oplus N(A)^{\perp}\oplus H_1\oplus H_1^{\perp}\to R(A)^{\perp}\oplus R(A)\oplus K_1\oplus K_1^{\perp},$$

where B_1 is an invertible operator from H_1 into K_1. Since A_1 is invertible, $M_C\in\Phi_+^-(H\oplus K)$ if and only if

$$\begin{pmatrix} 0 & C_{11} & C_{12}\\ 0 & B_1 & B_2\\ 0 & 0 & B_4\end{pmatrix}\in\Phi_+^-\big(N(A)\oplus H_1\oplus H_1^{\perp},R(A)^{\perp}\oplus K_1\oplus K_1^{\perp}\big).$$

Since

$$\begin{pmatrix} I & -C_{11}B_1^{-1} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} 0 & C_{11} & C_{12} \\ 0 & B_1 & B_2 \\ 0 & 0 & B_4 \end{pmatrix}\begin{pmatrix} I & 0 & 0 \\ 0 & I & -B_1^{-1}B_2 \\ 0 & 0 & I \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & -C_{11}B_1^{-1}B_2 + C_{12} \\ 0 & B_1 & 0 \\ 0 & 0 & B_4 \end{pmatrix}, \tag{4}$$

$M_C \in \Phi_+^-(H \oplus K)$ if and only if

$$\begin{pmatrix} 0 & 0 & -C_{11}B_1^{-1}B_2 + C_{12} \\ 0 & B_1 & 0 \\ 0 & 0 & B_4 \end{pmatrix} \in \Phi_+^-\big(N(A) \oplus H_1 \oplus H_1^\perp, R(A)^\perp \oplus K_1 \oplus K_1^\perp\big).$$

Since $d(A) = \infty$, define an operator $C_0 : \mathcal{K} \to \mathcal{H}$ by

$$C_0 = \begin{pmatrix} V_2 & V_2B_1^{-1}B_2 \\ 0 & V_1 \end{pmatrix} : H_1 \oplus H_1^\perp \to R(A)^\perp \oplus R(A),$$

where V_1 and V_2 are unitary operators. It is easy to show that C_0 is invertible. By Eq. (4), $M_{C_0} \in \Phi_+^-(H \oplus K)$ if and only if

$$M_0 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & B_4 \end{pmatrix} \in \Phi_+^-\big(N(A) \oplus H_1 \oplus H_1^\perp, R(A)^\perp \oplus K_1 \oplus K_1^\perp\big).$$

Thus $B_4 \in \Phi_+(H_1^\perp, K_1^\perp)$. It follows from $\left(\begin{smallmatrix} B_1 & 0 \\ 0 & B_4 \end{smallmatrix}\right) \in \Phi_+(K)$ that $R(B)$ is closed. This is a contradiction. Thus $R(B)$ is closed.

Since $n(M_0) = n(A) + n(B_4) < \infty$ and $d(M_0) = d(A) + d(B_4)$, we get that

$$n(M_0) - d(M_0) = n(A) - d(A) + n(B_4) - d(B_4) = \operatorname{ind}(A) + \operatorname{ind}(B),$$

the last equation follows from that B_1 is invertible. Thus $\operatorname{ind}(A) + \operatorname{ind}(B) \leqslant 0$. $\square$

Corollary 4.2. *For a given pair of operators* (A, B)*,*

$$\bigcup_{C \in \mathrm{Inv}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{ea}}(M_C) = \sigma_{\mathrm{ea}}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

In a similar way, we may obtain the next corollaries.

Corollary 4.3. *For given pair of operators* (A, B)*,*

$$\bigcup_{C \in \mathrm{Inv}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{SF}_-^+}(M_C) = \sigma_{\mathrm{SF}_-^+}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Corollary 4.4. *For given pair of operators* (A, B)*,*

$$\bigcup_{C \in \mathrm{Inv}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{w}}(M_C) = \sigma_{\mathrm{w}}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

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